Limit of sequence defined by linear 4-th degree Arithmetic Mean Recurrence.
Problem with a solution proposed by Arkady Alt , San Jose , California, USA

Let $(u_n)_{n\geq 0}$ be sequence defined recursively $u_{n+1} = \dfrac{u_n + u_{n-1} + u_{n-2} + u_{n-3}}{4}, n \geq 3.$

Determine $\lim\limits_{n\to\infty} u_n$.

As a variant.

Let $(u_n)_{n\geq 0}$ be sequence defined recursively

$$u_{n+1} = \frac{u_n + u_{n-1} + u_{n-2} + u_{n-3}}{4}, n \geq 3$$

Prove that $\lim\limits_{n\to\infty} u_n = \dfrac{4u_3 + 3u_2 + 2u_1 + u_0}{10}$

Solution.

We will find a, b, c and d such that $u_{n+1} + au_n + bu_{n-1} + cu_{n-2} = d, n \geq 2.$

We have $u_{n+1} + au_n + bu_{n-1} + cu_{n-2} - (u_n + au_{n-1} + bu_{n-2} + cu_{n-3}) = 0 \Leftrightarrow$
$u_{n+1} + (a-1)u_n + (b-a)u_{n-1} + (c-b)u_{n-2} - cu_{n-3} = 0.$ Comparing this recurrence with original we obtain that $a - 1 = b - a = c - b = -c = -\dfrac{1}{4} \Leftrightarrow c = \dfrac{1}{4}, b = \dfrac{1}{2}, a = \dfrac{3}{4}$ and, therefore, $d = u_3 + au_2 + bu_1 + cu_0 = \dfrac{4u_3 + 3u_2 + 2u_1 + u_0}{4}.$

Thus, $u_{n+1} + \dfrac{3}{4}u_n + \dfrac{2}{4}u_{n-1} + \dfrac{1}{4}u_{n-2} = \dfrac{4u_3 + 3u_2 + 2u_1 + u_0}{4}, n \geq 2.$

Let $t_n := u_n - \delta$ where δ can be determined by claim:

$t_{n+1} + \dfrac{3}{4}t_n + \dfrac{2}{4}t_{n-1} + \dfrac{1}{4}t_{n-2} = 0 \Leftrightarrow \delta + \dfrac{3}{4}\delta + \dfrac{2}{4}\delta + \dfrac{1}{4}\delta = d \Leftrightarrow \dfrac{5}{2}\delta = d \Leftrightarrow \delta = \dfrac{2d}{5} \Leftrightarrow$
$\delta = \dfrac{4u_3 + 3u_2 + 2u_1 + u_0}{10}$ •.

Thus, $u_n = \dfrac{4u_3 + 3u_2 + 2u_1 + u_0}{10} + t_n, n \in \mathbb{N} \cup \{0\},$ where t_n satisfy

(**1**) $\qquad t_{n+1} + \dfrac{3}{4}t_n + \dfrac{2}{4}t_{n-1} + \dfrac{1}{4}t_{n-2} = 0, n \geq 2.$

Let $P(x) := x^3 + \dfrac{3}{4}x^2 + \dfrac{1}{2}x + \dfrac{1}{4}$. Note that $P(x)$ is strictly increasing in $\mathbb{R}$. Indeed,
$P'(x) = 3x^2 + \dfrac{3}{2}x + \dfrac{1}{2} = 3\left(x + \dfrac{1}{4}\right)^2 + \dfrac{5}{16} > 0$ for all real x.

Since $P(-1) < 0$ and $P\left(-\dfrac{1}{4}\right) > 0$ then $P(x)$ has a unique real root $x_1 \in \left(-1, -\dfrac{1}{4}\right)$ and, therefore, $P(x) = (x - x_1)(x^2 + px + q) \Leftrightarrow x^3 + \dfrac{3}{4}x^2 + \dfrac{1}{2}x + \dfrac{1}{4} =$
$x^3 + (p - x_1)x^2 + (q - px_1)x - qx_1$ yield $p = x_1 + \dfrac{3}{4} \in \left(-\dfrac{1}{4}, \dfrac{1}{2}\right)$ and $q = -\dfrac{1}{4x_1} \in (0,1).$

Note that discriminant of quadratic trinomial $x^2 + px + q$ is negative, that is $p^2 < 4q$ (in particular it's yield $x_1^2 + px_1 + q > 0$).

Indeed,

$$p^2 - 4q = \left(x_1 + \frac{3}{4}\right)^2 - 4 \cdot \left(-\frac{1}{4x_1}\right) = \left(x_1 + \frac{3}{4}\right)^2 + \frac{1}{x_1} < \left(x_1 + \frac{3}{4}\right)^2 - 1 < \frac{1}{4} - 1.$$

Coming back to recurrence (**1**) we can rewrite it by two different way, namely, we have
$t_{n+2} + \dfrac{3}{4}t_{n+1} + \dfrac{1}{2}t_n + \dfrac{1}{4}t_{n-1} = t_{n+2} + (p - x_1)t_{n+1} + (q - px_1)t_n - qx_1t_{n-1} =$
$t_{n+2} + pt_{n+1} + qt_n - x_1(t_{n+1} + pt_n + qt_{n-1}) = t_{n+2} - x_1t_{n+1} + p(t_{n+1} - x_1t_n) + q(t_n - x_1t_{n-1}).$

Since $t_{n+2} + pt_{n+1} + qt_n = x_1(t_{n+1} + pt_n + qt_{n-1})$ then $t_{n+1} + tx_n + qt_{n-1} = x_1^{n-1}(t_2 + pt_1 + qt_0)$ and, therefore, $\alpha_n := t_{n+1} + pt_n + qt_{n-1}$ is infinitely small, that is $\lim\limits_{n\to\infty} \alpha_n = 0$ (because

$|x_1| < 1)$

Since $t_{n+2} - x_1 t_{n+1} + p(t_{n+1} - x_1 t_n) + q(t_n - x_1 t_{n-1}) = 0$ and $0 < q < 1,\ 4q > p^2$ then accordingly to **Lemma*** sequence $\beta_n := t_{n+1} - x_1 t_n$ infinitely small as well.

Indeed, let $\gamma_n = \dfrac{\beta_n}{\left(\sqrt{q}\right)^n}$ then $\beta_{n+2} + p\beta_{n+1} + q\beta_n = 0 \Leftrightarrow \gamma_{n+2} - 2 \cdot \dfrac{p}{-2\sqrt{q}}\gamma_{n+1} + \gamma_n = 0.$

Since $\left|\dfrac{p}{2\sqrt{q}}\right| < 1$ then γ_n is bounded and, therefore, $\lim\limits_{n\to\infty}\beta_n = \lim\limits_{n\to\infty}\left(\sqrt{q}\right)^n\gamma_n = 0.$

Thus, we have $t_{n+1} + pt_n + qt_{n-1} = \alpha_n,\ t_{n+1} - x_1 t_n = \beta_n.$

Hence, $(p + x_1)t_n + qt_{n-1} = \alpha_n - \beta_n$ and $t_n - x_1 t_{n-1} = \beta_{n-1}$ give us

$x_1(\alpha_n - \beta_n) + q\beta_{n-1} = x_1((p + x_1)t_n + qt_{n-1}) + q(t_n - x_1 t_{n-1}) \Leftrightarrow$

$(x_1^2 + px_1 + q)t_n = x_1(\alpha_n - \beta_n) + q\beta_{n-1} \Leftrightarrow t_n = \dfrac{x_1(\alpha_n - \beta_n) + q\beta_{n-1}}{x_1^2 + px_1 + q}$ and, therefore,

$\lim\limits_{n\to\infty} t_n = \dfrac{1}{x_1^2 + px_1 + q}\lim\limits_{n\to\infty}(x_1(\alpha_n - \beta_n) + q\beta_{n-1}) = 0.$

Hence, $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty}(t_n + \delta) = \delta = \dfrac{4u_3 + 3u_2 + 2u_1 + u_0}{10}.$

***Lemma**.

Any sequence $(x_n)_{n\ge0}$ such that $x_{n+1} - 2rx_n + x_{n-1} = 0, n \in \mathbb{N}$ and $0 < |r| < 1,$ is bounded.

Proof.

First we will find $(x_n)_{n\ge0}$ in supposition $|x_n| \le 1, n \in \mathbb{N} \cup \{0\}$. Let $\varphi := \cos^{-1}(r)$ and x_n can be represented as $x_n = \cos\varphi_n$, for some φ_n. Then we obtain

(1) $\cos\varphi_{n+1} - 2\cos\varphi \cdot \cos\varphi_n + \cos\varphi_{n-1} = 0 \Leftrightarrow \cos\varphi_{n+1} - \cos(\varphi_n + \varphi) = \cos(\varphi_n - \varphi) - \cos\varphi_{n-1}.$

Claim $\varphi_{n+1} = \varphi_n + \varphi, n \in \mathbb{N} \cup \{0\} \Leftrightarrow \varphi_n = n\varphi + \varphi_0,\ n \in \mathbb{N} \cup \{0\}$ give us the sequence $(\cos(n\varphi + \varphi_0))_{n\ge0}$ which satisfy (1) and, therefore, sequence $(\bar{x}_n)_{n\ge0} = (A\cos(n\varphi + \varphi_0))_{n\ge0}$ satisfy (1) as well for any given φ_0, A. For any sequence $(x_n)_{n\ge0}$ we can determine A and φ_0

by claim $A\cos\varphi_0 = x_0,\ A\cos(\varphi + \varphi_0) = x_1$. If $x_0 = 0$ then $\varphi_0 = \dfrac{\pi}{2}$ and $A = \dfrac{x_1}{\cos\left(\varphi + \frac{\pi}{2}\right)}$;

if $x_0 \ne 0$ then

$$\frac{\cos(\varphi + \varphi_0)}{\cos\varphi_0} = \frac{x_1}{x_0} \Leftrightarrow \cos\varphi - \sin\varphi \cdot \tan\varphi_0 = \frac{x_1}{x_0} \Leftrightarrow \varphi_0 = \tan^{-1}\left(\cot\varphi - \frac{x_1}{x_0\sin\varphi}\right)$$

and then $A = \dfrac{x_0}{\cos\varphi_0}$.

Thus, we have two sequences $(x_n)_{n\ge0}$ and $(\bar{x}_n)$ both satisfy $x_{n+1} - 2rx_n + x_{n-1} = 0, n \in \mathbb{N}$ and $x_0 = \bar{x}_0, x_1 = \bar{x}_1$ and, therefore, by math Induction we obtain $x_n = \bar{x}_n, n \in \mathbb{N} \cup \{0\}$.

So, $|x_n| \le A, n \in \mathbb{N} \cup \{0\}$. ■